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NOTE ON THE STABILITY OF MOTION IN A CIRCLE UNDER A CENTRAL FORCE

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It is well known that, when a particle under a central force $f(r)$ per unit mass, circular orbits are possible when the force is attractive. If the circular orbit be of radius a, the condition usually given for stability of this orbit (1) under radial perturbation is

$$af'(a) + 3f(a) < 0. \tag{1}$$

This criterion for stability assumes that $f(r)$ is differentiable at $r = a$. It does not seem to have been realised that it is possible, by means of Liapunov Theory to obtain a stability criterion which is valid when $f(r)$ is merely integrable in some interval including $r = a$. The process is as follows.
With the usual notation

$$\ddot{r} - r\dot{\theta}^2 = f(r)$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

whence $r^2\dot{\theta} = h$ a constant.
Thus

$$\ddot{r} - \frac{h^2}{r^3} = f(r) \tag{2}$$

and the condition that a circular orbit $r = a$ exists is

$$h^2 + a^3 f(a) = 0 \tag{3}$$

If there is a radial perturbation $r = a + \rho$, equation (2) becomes

$$\ddot{\rho} - \frac{h^2}{(a+\rho)^3} - f(a+\rho) = 0. \tag{4}$$

Equation (4) can be rewritten in the form

$$\dot{\rho} = \sigma \tag{5a}$$

$$\dot{\sigma} = \frac{h^2}{(a+\rho)^3} + f(a+\rho) \tag{5b}$$

Consider now the Liapunov Function (2)

$$V(\sigma,\rho) = \frac{1}{2}\sigma^2 - \int_0^\rho \left[\frac{h^2}{(a+u)^3} + f(a+u)\right] du. \tag{6}$$

It can easily be shown that

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$$\frac{dV}{dt} = \dot{\sigma}\frac{\partial V}{\partial \sigma} + \dot{\rho}\frac{\partial V}{\partial \rho} = 0.$$

Thus, if V as defined by the expression (6) is positive definite, it is a weak Liapunov Function and the motion is uniformly, but not asymptotically, stable.
The condition for V to be positive definite will be

$$\left[\frac{h^2}{(a+\rho)^3} + f(a+\rho)\right]\rho < 0. \tag{7}$$

Using condition (3), the inequality (7) can be written as

$$\left[-\frac{a^3 f(a)}{(a+\rho)^3} + f(a+\rho)\right]\rho < 0$$

or

$$[a^3 f(a) - (a+\rho)^3 f(a+\rho)]\rho > 0 \tag{8}$$

and this is to hold in some interval including $\rho = 0$.
If ρ is initially within this interval, it will remain so. It can easily be seen that this condition, which holds under a broader set of conditions on f, reduces to the usual condition (1) when f is differentiable.

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